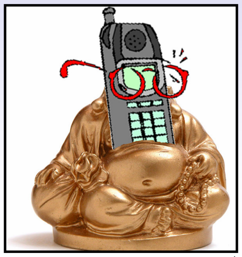
Equilibrium Concepts

Nash Equilibria, Mixed Strategy Equilbria, Coalitional Games, the Core, Shapley Value, Nash Bargaining,



Steady-states

- Recall model of <N,A,{d_i},T> which we characterize with the evolution function d
- Steady-state is a point where $a^* = d(a^*)$ for all $t \geq t^*$
- Obvious solution: solve for fixed points of d.
- For non-cooperative radios, if a^* is a fixed point under synchronous timing, then it is under the other three timings.
- Works well for convex action spaces
  - Not always guaranteed to exist
  - Value of fixed point theorems
- Not so well for finite spaces
  - Generally requires exhaustive search

Nash Equilibrium

"A steady-state where each player holds a correct expectation of the other players' behavior and acts rationally." - Osborne

An action vector from which no player can profitably unilaterally deviate.

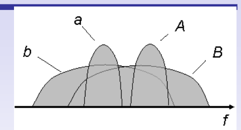
Definition

An action tuple a is a NE if for every $i \in N$ $u_i(a_i, a_{-i}) \geq u_i(b_i, a_{-i})$ for all $b_i \in A_i$.

Note showing that a point is a NE says nothing about the process by which the steady state is reached. Nor anything about its uniqueness.
Also note that we are implicitly assuming that only pure strategies are possible in this case.

Examples

- Cognitive Radios' Dilemma
  - Two radios have two signals to choose between {n,w} and {N,W}
  - n and N do not overlap
  - Higher throughput from operating as a high power wideband signal when other is narrowband



Γ	N	W
n	(9.6,9.6)	(3.2,21)
w	(21,3.2)	(7,7)

- Jamming Avoidance
  - Two channels
  - No NE

Jammer

Transmitter	0	1
0	(-1,1)	(1,-1)
1	(1,-1)	(-1,1)

How do the players find the Nash Equilibrium?

- Preplay Communication
  - Before the game, discuss their options. Note only NE are suitable candidates for coordination as one player could profitably violate any agreement.
- Rational Introspection
  - Based on what each player knows about the other players, reason what the other players would do in its own best interest. (Best Response - tomorrow) Points where everyone would be playing "correctly" are the NE.
- Focal Point
  - Some distinguishing characteristic of the tuple causes it to stand out. The NE stands out because it's every player's best response.
- Trial and Error
  - Starting on some tuple which is not a NE a player "discovers" that deviating improves its payoff. This continues until no player can improve by deviating. Only guaranteed to work for Potential Games (couple weeks)

Nash Equilibrium as a Fixed Point

- Individual Best Response

$$\hat{B}_i(a) = \{b_i \in A_i : u_i(b_i, a_{-i}) \geq u_i(a_i, a_{-i}) \forall a_i \in A_i\}$$

- Synchronous Best Response

$$\hat{B}(a) = \times_{i \in N} \hat{B}_i(a)$$

- Nash Equilibrium as a fixed point

$$a^* = \hat{B}(a^*)$$

- Fixed point theorems can be used to establish existence of NE (see dissertation)
- NE can be solved by implied system of equations

Example solution for Fixed Point by Solving for Best Response Fixed Point

- Bandwidth Allocation Game
  - Five cognitive radios with each radio, *i*, free to determine the number of simultaneous frequency hopping channels the radio implements, $c_i \in [0,\infty)$.
  - Goal $u_i(c) = P(c)c_i - C_i(c_i)$
  - $P(c)$ fraction of symbols that are not interfered with (making $P(c)c_i$ the goodput for radio *i*)
  - $C_i(c_i)$ is radio *i*'s cost for supporting c_i simultaneous channels.

$$u_i(c) = \left(B - \sum_{k\in N} c_k\right)c_i - Kc_i$$

Best Response Analysis

Goal $u_i(c) = \left(B - \sum_{k\in N} c_k\right)c_i - Kc_i$

Best Response $c_i = \hat{B}_i(c) = \left(B - K - \sum_{k\in N\setminus i} c_k\right)/2$

Simultaneous System of Equations

c_1	+	$0.5\,c_2$	+	$0.5\,c_3$	+	$0.5\,c_4$	+	$0.5\,c_5$	=	$(B-K)/2$
$0.5\,c_1$	+	c_2	+	$0.5\,c_3$	+	$0.5\,c_4$	+	$0.5\,c_5$	=	$(B-K)/2$
$0.5\,c_1$	+	$0.5\,c_2$	+	c_3	+	$0.5\,c_4$	+	$0.5\,c_5$	=	$(B-K)/2$
$0.5\,c_1$	+	$0.5\,c_2$	+	$0.5\,c_3$	+	c_4	+	$0.5\,c_5$	=	$(B-K)/2$
$0.5\,c_1$	+	$0.5\,c_2$	+	$0.5\,c_3$	+	$0.5\,c_4$	+	c_5	=	$(B-K)/2$

Solution $\hat{c}_i = (B-K)/6 \;\forall i \in N$

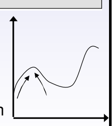
Generalization $\hat{c}_i = (B-K)/(|N|+1) \;\forall i \in N$

Significance of NE for CRNs

Theorem 4.1: NE and Cognitive Radio Network Steady States (*)
Given cognitive radio network $\langle N, A, \{u_i\}, \{d_i\}, T\rangle$ where all players are autonomously rational, if the game $\langle N, A, \{u_i\}\rangle$ has an NE a^*, then a^* is a fixed point for *d*.

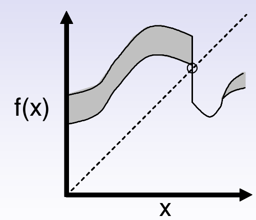
Proof: Suppose a^* is not a fixed point. Then for some $i \in N$, there must be some $b_i \in d_i(a^*)$ with $b_i \neq a_i^*$ such that $u_i(b_i, a_{-i}^*) > u_i(a_i^*, a_{-i}^*)$. But this contradicts the assumption that a^* is an NE. Therefore, a^* must be a fixed point for *d*.

- Why not "if and only if"?
  - Consider a self-motivated game with a local maximum and a hill-climbing algorithm.
  - For many decision rules, NE do capture all fixed points (see dissertation)
- Identifies steady-states for all "intelligent" decision rules with the same goal.
- Implies a mechanism for policy design while accommodating differing implementations
  - Verify goals result in desired performance
  - Verify radios act intelligently



Key Theorem for NE Existence

- Kakutani's Fixed Point Theorem
  - Let $f: X \rightarrow X$ be an upper semi-continuous convex valued correspondence from a non-empty compact convex set $X \subset \mathbb{R}^n$, then there is some $x^* \in X$ such that $x^* \in f(x^*)$



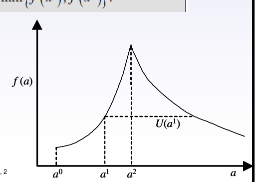
Nash Equilibrium Existence

Theorem 4.3: *Glicksberg-Fan-Debreu Fixed Point Theorem* [Fudenberg_91]
Given normal form game $\Gamma = \langle N, A, \{u_i\}\rangle$ where A_i are nonempty compact convex subsets of $\mathbb{R}^m \;\forall i \in N$. If $\forall i \in N$ u_i is continuous in *a* and quasi-concave in a_i then Γ has a pure strategy NE.

Definition 4.7: *Quasi-concavity*
A function $f: X \rightarrow \mathbb{R}$ is said to be *quasi-concave* if $\forall x^1, x^2 \in X$, $\alpha \in (0,1)$ the following relationship is satisfied: $f(\alpha x^1 + (1-\alpha)x^2) \geq \min\{f(x^1), f(x^2)\}$.

Visualizable Definition of Quasi-Concavity
All upper-level sets are convex

$$U(a^*) = \{a \in A : f(a) \geq a^*\}$$



My Favorite Mixed Strategy Story

Pure Strategies in an Extended Game
Consider an extensive form game where each stage is a strategic form game and the action space remains the same at each stage. Before play begins, each player chooses a probabilistic strategy that assigns a probability to each action in his action set. At each stage, the player chooses an action from his action set according to the probabilities he assigned before play began.

Example
Consider a video football game which will be simulated. Before the game begins two players assign probabilities of calling running plays or passing plays for both offense or defense. In the simulation, for each down the kind of play chosen by each team is based on the initial probabilities assigned to kinds of plays. (Play NCAA2003)

Example Mixed Strategy Game

Jamming game

		q	$(1\text{-}q)$
		a_2	b_2
p	a_1	1,-1	-1, 1
$(1\text{-}p)$	b_1	-1, 1	1, -1

Action Tuples Probabilities

(a_1,a_2)	pq
(a_1,b_2)	$p(1\text{-}q)$
(b_1,a_2)	$(1\text{-}p)q$
(b_1,b_2)	$(1\text{-}p)(1\text{-}q)$

Expected Utilities

$$U_1(p,q) = pq(1) + p(1-q)(-1) + \cdots + (1-p)q(-1) + (1-p)(1-q)(1)$$

$$U_2(p,q) = pq(-1) + p(1-q)(1) + \cdots + (1-p)q(1) + (1-p)(1-q)(-1)$$

Sets of probability distributions

$\Delta(A_1)=\{p,(1\text{-}p): \forall p\in[0,1]\}$

$\Delta(A_2)=\{q,(1\text{-}q): \forall q\in[0,1]\}$

Nash Equilibrium in a Mixed Strategy Game

Definition Mixed Strategy Nash Equilibrium

A mixed strategy profile α^* is a NE iff $\forall i\in N$

$$U_i(\alpha_i^*, \alpha_{-i}^*) \geq U_i(\beta_i, \alpha_{-i}^*) \forall \beta_i \in \Delta(A_i)$$

Best Response Correspondence

$$BR_i(\alpha_{-i}) = \arg \max_{\alpha_i \in \Delta(A_i)} U_i(\alpha_i, \alpha_{-i})$$

Alternate NE Definition

Consider $B(\alpha) = \times_{i\in N} BR_i(\alpha)$

A mixed strategy profile α^* is a NE iff

$$\alpha^* \in B(\alpha^*)$$

Nash Equilibrium

$$U_1(p,q) = pq(1) + (1-p)(1-q)(3)$$

$$U_2(p,q) = pq(1) + (1-p)(1-q)(3)$$

Best Response

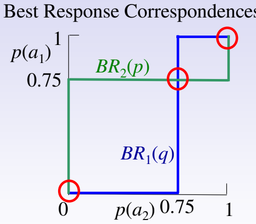
$$\frac{\partial u_1}{\partial p}(q) = 4q - 3$$

$$\frac{\partial u_2}{\partial q}(p) = 4p - 3$$

$$BR_1(q) = \begin{cases} 1 & q > 3/4 \\ [0,1] & q = 3/4 \\ 0 & q < 3/4 \end{cases}$$

$$BR_2(p) = \begin{cases} 1 & p > 3/4 \\ [0,1] & p = 3/4 \\ 0 & p < 3/4 \end{cases}$$

Best Response Correspondences



Note: both NE of the strategic game are NE in its mixed extension

Interesting Properties of Mixed Strategy Games

1. Every Mixed Extension of a Strategic Game has an NE.
2. A mixed strategy α_i is a best response to α_{-i} iff every action in the support of α_i is itself a best response to α_{-i}.
3. Every action in the support of any player's equilibrium mixed strategy yields the same payoff to that player.

Coalitional Game (with transferable payoff)

- Concept: groups of players (called coalitions) conspire together to implement actions which yields a result for the coalition. The value received by the coalition is then distributed among the coalition members.
- Where do radios collaborate and distribute value?
  - 802.16h interference groups – allocation of bandwidth
  - Distribution of frequencies/spreading codes among cells
  - File sharing in P2P network

$$v: 2^N \setminus \varnothing \to \mathbb{R}$$

- Transferable utility refers to existence of some commodity for which a player's utility increases by one unit for every unit of the commodity it receives
- Game Components, $\langle N,v\rangle$
  - N set of players
  - Characteristic function
  - Coalition, $S\subseteq N$
- How is this value distributed?
  - Payoff vector, $(x_i)_{i\in S}$

$$x(S) = \sum_{i\in S} x_i$$

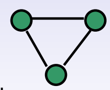
- Payoff vector is said to be S-feasible if $x(S) \leq v(S)$

The Core (Transferrable)

- **<u>The Core</u>**
  - For $\langle N,v\rangle$, the set of feasible payoff profiles, $(x_i)_{i\in S}$ for which there is no coalition S and S-feasible payoff vector $(y_i)_{i\in S}$ for which $y_i > x_i$ for all $i\in S$.
- General principles of the NE also apply to the Core:
  - Number of solutions for a game may be anywhere from 0 to ∞
  - May be stable or unstable.

Example

- Suppose three radios, $N = \{1,2,3\}$, can choose to participate in a peer-to-peer network.
- Characteristic Function
  - $v(N) = 1$
  - $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = \alpha \in [0,1]$
  - $v(1) = v(2) = v(3) = 0$



- Loosely, α indicates # of duplicated files
- If $\alpha > 2/3$, Core is empty

Example adaptations for $\alpha = 4/5$: x = (2/5, 2/5,0) x = (0, 3/5, 1/5) x = (2/5, 0, 2/5) x = (1/3, 1/3, 1/3) x = (2/5, 2/5,0)

Comments on the Core

- Possibility of empty core implies that even when radios can freely negotiate and form arbitrary coalitions, no steady-state may exist
- Frequently very large (infinite) number of steady-states, e.g., $\alpha < 2/3$
  - Makes it impossible to predict exact behavior
- Existence conditions for the Core, but would need to cover some linear programming concepts
- Related (but not addressed today) concepts:
  - Bargaining Sets, Kernel, Nucleolus

Strong NE

- Concept: Assume radios are able to collaborate, but utilities aren't necessarily transferrable
- An action tuple a^* such that

$$u_i\left(a^*\right) \geq u_i\left(a_S, a_{-S}^*\right) \; \forall S \subseteq N, a_S \in \underset{i \in S}{\times} A_i$$

No Strong NE

Γ	N	W
n	(9.6,9.6)	(3.2,21)
w	(21,3.2)	(7,7)

Unique Strong NE

	N	W
n	(9.6,9.6)	(9.6, 21)
w	(21, 9.6)	(22, 22)

Motivation for Shapley value

- Core was generally either empty or very large. Want a "good" single solution.
- Terminology
  - Marginal Contribution of i

    $$\Delta_i(S) = v(S \cup i) - v(S)$$
  - Interchangeability of i, j

    $$\Delta_i(S) = \Delta_j(S) \, \forall S \subseteq N \setminus \{i, j\}$$
  - Dummy player (no synergy)

    $$\Delta_i(S) = v(\{i\}) \, \forall S \subseteq N \setminus i$$

Axioms for Shapley Value

- Let ψ be some distribution of value for a TU coalition game
- Symmetry:
  - If i and j are interchangeable, then $\psi_i(v) = \psi_j(v)$
- Dummy:
  - If i is a dummy, then $\psi_i(v) = v(\{i\})$
- Additivity:
  - Given $\langle N,v \rangle$ and $\langle N,w \rangle$, $\psi_i(v + w) = \psi_i(v) + \psi_i(w)$ for all $i \in N$, where $v+w = v(S) + w(S)$
- Balanced Contributions
  - Given $\langle N,v \rangle$, $\psi_i(N,v) - \psi_i\left(N \setminus j.v^{N \setminus j}\right) = \psi_j(N,v) - \psi_j\left(N \setminus i.v^{N \setminus i}\right)$

Shapley Value

$$\psi_i(S) = \sum_{S \subseteq N \setminus i} \frac{|S|!\left(|N| - |S| - 1\right)!}{|N|!} \left(v(S \cup i) - v(S)\right)$$

Marginal Value Contributed by i

Probability that i will be next one invited to the grand coalition given that coalition S is already part of the coalition assuming random ordering.

Only assignment (value) that satisfies balanced contributions; only assignment that simultaneously satisfies symmetry, dummy, and additivity axioms

Implications of Shapley Value

- One form of a fair allocation
  - What you receive is based on the value you add
  - Independent of order of arrival
  - I liken it to setting salaries according to the Value Over Replacement Player concept in Baseball Statistics
- "Better" solution concept than the core as it's a single payoff as opposed to a potentially infinite number
- Allows for analysis of relative "power" of different players in the system

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Bargaining Problem

- Components: $\langle F, v \rangle$
  - Feasible payoffs *F*, closed convex subset of $\mathbb{R}^n$
  - Disagreement Point $v = (v_1, v_2)$
    - What 1 or 2 could achieve without bargaining
- Example:
  - Even if system is jammed, still gets some throughput
  - Member of 802.16h interference group and try its luck
- *F* is said to be *essential* if there is some $y \in F$ such that $y_1 > v_1$ and $y_2 > v_2$
- If contracts are "binding" then *F* could be the payoffs corresponding to entire original action space
- Otherwise, *F* may need to be drawn from the set of NE or from enforceable set (see punishment in repeated games)
- A particular solution is referred to by $\phi(F, v) \in \mathbb{R}^n$

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Desirable Bargaining Axioms about a Solution

- Strong Efficiency
  - $\phi(F, v)$ is Pareto Efficient
- Individually Rational
  - $\phi(F, v) \geq v$
- Scale Covariance
  - For any $\lambda_1, \lambda_2, \gamma_1, \gamma_2 \in \mathbb{R}$, $\lambda_1, \lambda_2 > 0$, if

$$G = \left\{ \left(\lambda_1 x_1 + \gamma_1, \lambda_2 x_2 + \gamma_2 \right) \mid \left(x_1, x_2 \right) \in F \right\}$$

then

$$\phi(G, w) = \left(\lambda_1 \phi_1(F, v) + \gamma_1, \lambda_2 \phi_2(F, v) + \gamma_2 \right)$$

$$w = \left(\lambda_1 v_1 + \gamma_1, \lambda_2 v_2 + \gamma_2 \right)$$

- Independence of Irrelevant Alternatives
  - If $G \subseteq F$ and *G* is closed and convex and $\phi(F, v) \in G$, then $\phi(G, v) = \phi(F, v)$
- Symmetry
  - If $v_1 = v_2$ and $\{(x_1, x_2) \mid (x_2, x_1) \in F\} = F$, then $\phi_1(F, v) = \phi_2(F, v)$

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Nash Bargaining Solution

- NBS

$$\phi(F, v) \in \arg \max_{x \in F, x \geq v} \prod_{i \in N} \left(x_i - v_i \right)$$

- Interestingly, this is the only bargaining solution which simultaneously satisfies the preceding 5 axioms

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GT framework for BW allocation [Yaiche]: System Model

- *N* users
- *L* links
- Users compete for the total link capacity
- Each user has a minimum rate MR_i and peak rate PR_i
- Admissible rate vector is given by,

$$X_0 = \left\{ x \in \mathbb{R}^N \mid x \geq MR, x \leq PR, \text{and } Ax \leq C \right\}$$

C : vector of link capacities

A_{L^*N}: $a_{lp} = 1$ if link belongs to path *p*, else 0.

Scenario given in H. Yaiche, R. Mazumdar, C. Rosenberg, "A game theoretic framework for bandwidth allocation and pricing in broadband networks", IEEE/ACM Transactions on Networking, Volume: 8 , Issue: 5 , Oct. 2000, pp. 667-678.

Centralized Optimization Problem

- $$Max_{\{x\}} \prod_{i=1}^{N} \left(x_i - MR_i \right)$$

$$\begin{aligned} st: \; & x_i \geq MR_i \quad i \in \{1 \ldots N\} && (v) \\ & x_i \leq PR_i \quad i \in \{1 \ldots N\} && (F) \\ & (Ax)_l \geq (C)_l \quad l \in \{1 \ldots L\} && (F) \end{aligned}$$

- Unique NBS exists

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Steady-State Summary

- Not every game has a steady-state
- NE are analogous to fixed points of self-interested decision processes
- NE can be applied to procedural and ontological radios
  - Don't need to know decision rule, only goals, actions, and assumption that radios act in their own interest
- A game (network) may have 0, 1, or many steady-states
- All finite normal form games have an NE in its mixed extension
  - Over multiple iterations, implies constant adaptation
- More complex game models yield more complex steady-state concepts
- Can define steady-states concepts for coalitional games
  - Frequently so broad that specific solutions are used

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